MATHEMATICS

SUMMATION OF SOME SERIES OF BESSEL FUNCTIONS

BY

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1. The following formulas due to Lommel

$$(\tfrac{1}{2}x)\, J_\nu(x) = \sum_{n=0}^{\infty} (-1)^n (\nu+2n+1)\, J_{\nu+2n+1}(x),$$

$$\tfrac{1}{2}x \sin x = \sum_{n=1}^{\infty} (-1)^n (2n)^2\, J_{2n}(x),$$

$$\tfrac{1}{2}x \cos x = \sum_{n=0}^{\infty} (-1)^n (2n+1)^2\, J_{2n+1}(x)$$

are well known. In attempting to generalize these results RUTGERS [3], [4] has summed a great many series of the type

$$\sum_{n=0}^{\infty} (-1)^n (\nu+2n)^k\, J_{\nu+2n}(x), \tag{1.1}$$

where k is a non-negative integer. He showed for example that

$$\sum_{n=0}^{\infty} (-1)^n (2n+1)^k\, J_{2n+1}(x) = A_k(x) \cos x + B_k(x) \sin x,$$

where $A_k(x)$, $B_k(x)$ are polynomials in x. Similarly

$$\sum_{n=0}^{\infty} (-1)^n (\nu+2n)^{4k+1}\, J_{\nu+2n}(x) = A_k^{(\nu)}(x)\, I_\nu(x) + B_k^{(\nu)}(x)\, I_{\nu-1}(x),$$

where $A_k^{(\nu)}(x)$, $B_k^{(\nu)}(x)$ are polynomials in x. The coefficients of these polynomials are expressed in terms of multiple sums involving binomial coefficients. We remark that in [3], [4] RUTGERS has given explicit results for a number of small values of k.

In the present paper we shall put Rutgers' results in a more explicit form and at the same time point out how certain functions that are of importance in finite differences occur naturally in the present context. The main results of the paper are contained in formulas (2.7), (2.8), (3.4), (4.5) and (5.3) below.

2. Making use of the familiar expansion

$$(\tfrac{1}{2}x)^\nu = \sum_{n=0}^{\infty} (\nu+2n) \frac{\Gamma(\nu+n)}{n!} J_{\nu+2n}(x),$$

we have

$$(\tfrac{1}{2}x)^\nu \sin x = \sum_{r=0}^{\infty}(-1)^r \frac{2^{2r+1}}{(2r+1)!}(\tfrac{1}{2}x)^{\nu 2r+1}$$

$$= \sum_{r=0}^{\infty}(-1)^r \frac{2^{2r+1}}{(2r+1)!}\sum_{s=0}^{\infty}(\nu+2r+2s+1)\frac{\Gamma(\nu+2r+s+1)}{s!}J_{\nu+2r+2s+1}(x)$$

$$= \sum_{n=0}^{\infty}(\nu+2n+1)J_{\nu+2n+1}(x)\cdot\sum_{r=0}^{n}(-1)^r\frac{\Gamma(\nu+n+r+1)}{(2r+1)!(n-r)!}2^{2r+1}$$

$$= \sum_{n=0}^{\infty}\frac{(\nu+2n+1)\,\Gamma(\nu+n+1)}{n!}\sum_{r=0}^{n}\frac{(-n)_r\,(\nu+n+1)}{(2r+1)!}2^{2r+1}.$$

Since

$$\sum_{r=0}^{n}\frac{(-n)_r(\nu+n+1)_r}{(2r+1)!}2^{2r+1} = 2\sum_{r=0}^{n}\frac{(-n)_r(\nu+n+1)_r}{r!(3/2)_r} = 2\frac{(\frac{1}{2}-\nu-n)_n}{(3/2)_n}$$

$$= (-1)^n 2\frac{(\nu+\frac{1}{2})_n}{(3/2)_n},$$

it follows that

$$(2.1)\quad \begin{cases} (\tfrac{1}{2}x)^\nu \sin x = 2\Gamma(\nu+1)\displaystyle\sum_{n=0}^{\infty}(-1)^n(\nu+2n+1)\frac{(\nu+1)_n(\nu+\frac{1}{2})_n}{n!(3/2)_n}J_{\nu+2n+1}(x) \\ \qquad = 2\Gamma(\nu+1)\displaystyle\sum_{n=0}^{\infty}(-1)^n(\nu+2n+1)\frac{(2\nu+1)_{2n}}{(2n+1)!}J_{\nu+2n+1}(x). \end{cases}$$

Similarly we find that

$$(2.2)\qquad (\tfrac{1}{2}x)^\nu \cos x = \Gamma(\nu)\sum_{n=0}^{\infty}(-1)^n(\nu+2n)\frac{(2\nu)_{2n}}{(2n)!}J_{\nu+2n}(x).$$

In particular if we take $\nu=k$ we get

$$(2.3)\qquad (\tfrac{1}{2}x)^k \sin x = \frac{(k-1)!}{(2k-1)!}\sum_{n=0}^{\infty}(-1)^n(k+2n+1)(2n+2)_{2k-1}J_{k+2n+1}(x),$$

$$(2.4)\qquad (\tfrac{1}{2}x)^k \cos x = \frac{(k-1)!}{(2k-1)!}\sum_{n=0}^{\infty}(-1)^n(k+2n)(2n+1)_{2k-1}J_{k+2n}(x).$$

Now consider

$$\sum_s(-1)^s a_{k,2s}\frac{(4s-1)!}{(2s-1)!}(\tfrac{1}{2}x)^{2s}\cos x + \sum_s(-1)^s a_{k,2s-1}\frac{(4s-3)!}{(2s-2)!}(\tfrac{1}{2}x)\sin x$$

$$= \sum_s(-1)^s a_{k,2s}\sum_{n=0}^{\infty}(-1)^n\,(2n+2s)\,(2n+2)_{4s-1}\,J_{2n+2s}(x)$$

$$+ \sum_s(-1)^s a_{k,2s-1}\sum_{n=0}^{\infty}(-1)^n\,(2n+2s)\,(2n+1)_{4s-3}\,J_{2n+2s}(x)$$

$$= \sum_s a_{k,2s}\sum_{n=s}^{\infty}(-1)^n\,(2n)\,(2n-2s+1)_{4s-1}\,J_{2n}(x)$$

$$+ \sum_s a_{k,2s-1}\sum_{n=s}^{\infty}(-1)^n\,(2n)\,(2n-2s+2)_{4s-3}\,J_{2n}(x)$$

$$= \sum_{n=0}^{\infty}(-1)^n J_{2n}(x)\sum_s a_{k,s}\,(2n)\,(2n-s+1)_{2s-1}.$$

The coefficients $a_{k,s}$ are at our disposal. We choose them so that

$$x^{2k}=\sum_{s=1}^{k} a_{k,s}\, x(x-s+1)_{2s-1}. \tag{2.5}$$

By the Stirling interpolation formula (see for example [2], p. 203) it follows that

$$a_{k,s}=\frac{1}{(2s)!}\,[\Delta^{2s}x^{2k}]_{x=-s}=\frac{1}{(2s)!}\sum_{j=0}^{2s}(-1)^j(2sj)(j-s)^{2k}. \tag{2.6}$$

We have therefore proved that for $k \geqslant 1$

$$\left\{\begin{aligned}\sum_{n=1}^{\infty}(-1)^n(2n)^{2k}J_{2n}(x)=\tfrac{1}{2}\sum_{s=1}^{[k/2]}(-1)^s a_{k,2s}\frac{(4s)!}{(2s)!}(\tfrac{1}{2}x)^{2s}\cos x\\ +\tfrac{1}{2}\sum_{s=1}^{[(k-1)/2]}(-1)^s a_{k,2s-1}\frac{(4s-2)!}{(2s-1)!}(\tfrac{1}{2}x)^{2s-1}\sin x,\end{aligned}\right. \tag{2.7}$$

where the coefficients $a_{k,s}$ are defined by (2.6).

In the second place we have

$$\begin{aligned}\sum_s(-1)^s a_{k,2s}\frac{(4s-1)!}{(2s-1)!}(\tfrac{1}{2}x)^{2s}\sin x+\sum_s(-1)^s a_{k,2s+1}\frac{(4s+1)!}{(2s)!}(\tfrac{1}{2}x)^{2s+1}\cos x\\ =\sum_s(-1)^s a_{k,2s}\sum_{n=0}^{\infty}(-1)^n\,(2n+2s+1)\,(2n+2)_{4s-1}\,J_{2n+2s+1}(x)\\ +\sum_s(-1)^s a_{k,2s+1}\sum_{n=0}^{\infty}(-1)^n\,(2n+2s+1)\,(2n+1)_{4s+1}\,J_{2n+2s+1}(x)\\ =\sum_s a_{k,2s}\sum_{n=s}^{\infty}(-1)^n\,(2n+1)\,(2n-2s+2)_{4s-1}\,J_{2n+1}(x)\\ +\sum_s a_{k,2s+1}\sum_{n=s}^{\infty}(-1)^n\,(2n+1)\,(2n-2s+1)_{4s+1}\,J_{2n+1}(x)\\ =\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)\sum_s a_{k,s}\,(2n+1)\,(2n-s+2)_{2s-1}\\ =\sum_{n=0}^{\infty}(-1)^n\,(2n+1)^{2k}\,J_{2n+1}(x).\end{aligned}$$

We have therefore for $k \geqslant 1$

$$\left\{\begin{aligned}\sum_{n=0}^{\infty}(-1)^n(2n+1)^{2k}J_{2n+1}(x)=\tfrac{1}{2}\sum_{s=1}^{[k/2]}(-1)^s a_{k,2s}\frac{(4s)!}{(2s)!}(\tfrac{1}{2}x)^{2s}\sin x\\ +\tfrac{1}{2}\sum_{s=0}^{[(k-1)/2]}(-1)^s a_{k,2s+1}\frac{(4s+2)!}{(2s+1)!}(\tfrac{1}{2}x)^{2s+1}\cos x.\end{aligned}\right. \tag{2.8}$$

These results may be compared with VIIa, VIIb, VIIIa, VIIIb of [4].

3. If we replace x by ix in (2.7) and (2.8) we get

$$\left\{\begin{aligned}\sum_{n=1}^{\infty}(2n)^{2k}I_{2n}(x)=\tfrac{1}{2}\sum_{s=1}^{[k/2]}a_{k,2s}\frac{(4s)!}{(2s)!}(\tfrac{1}{2}x)^{2s}\cosh x\\ +\tfrac{1}{2}\sum_{s=1}^{[(k+1)/2]}a_{k,2s-1}\frac{(4s-2)!}{(2s-1)!}(\tfrac{1}{2}x)^{2s-1}\sinh x,\end{aligned}\right. \tag{3.1}$$

$$(3.2)\quad \begin{cases} \sum_{n=0}^{\infty}(2n+1)^{2k}I_{2n+1}(x)=\frac{1}{2}\sum_{s=1}^{[k/2]}a_{k,2s}\frac{(4s)!}{(2s)!}(\frac{1}{2}x)^{2s}\sinh x \\ \qquad +\frac{1}{2}\sum_{s=0}^{[(k-1)/2]}a_{k,2s+1}\frac{(4s+2)!}{(2s+1)!}(\frac{1}{2}x)^{2s+1}\cosh x. \end{cases}$$

Put

$$(3.3)\qquad \varphi_k(x)=\sum_{s=1}^{k}a_{k,s}\frac{(2s)!}{s!}(\tfrac{1}{2}x)^s.$$

Then it follows immediately from (3.1) and (3.2) that

$$(3.4)\qquad \sum_{n=1}^{\infty}n^{2k}I_n(x)=\tfrac{1}{2}e^x\varphi_k(x);$$

moreover (3.4) implies both (3.1) and (3.2).

4. Returning to (2.1) and (2.2) we take $\nu=k\mp\frac{1}{2}$ so that

$$(4.1)\quad \begin{cases} (\frac{1}{2}x)^{k-\frac{1}{2}}\sin x \\ \qquad =\frac{\pi^{\frac{1}{2}}}{2^{2k-1}(k-1)!}\sum_{n=0}^{\infty}(-1)^n(2n+k+\frac{1}{2})(2n+2)_{2k-2}J_{2n+k+\frac{1}{2}}(x), \end{cases}$$

$$(4.2)\quad \begin{cases} (\frac{1}{2}x)^{k+\frac{1}{2}}\cos x \\ \qquad =\frac{\pi^{\frac{1}{2}}}{2^{2k}k!}\sum_{n=0}^{\infty}(-1)^n(2n+k+\frac{1}{2})(2n+1)_{2k}J_{2n+k+\frac{1}{2}}(x). \end{cases}$$

Then

$$\begin{aligned} &\pi^{-\frac{1}{2}}\sum_s(-1)^s\,2^{4s-1}(2s-1)!\,b_{k,2s-1}\,(\tfrac{1}{2}x)^{2s-\frac{1}{2}}\sin x \\ &\qquad +\pi^{-\frac{1}{2}}\sum_s(-1)^s\,2^{4s}\,(2s)!\,b_{k,\,2s}(\tfrac{1}{2}x)^{2s+\frac{1}{2}}\cos x \\ &=\sum_s(-1)^s\,b_{k,2s-1}\sum_{n=0}^{\infty}(-1)^n\,(2n+2s+\tfrac{1}{2})\,(2n+2)_{4s-2}\,J_{2n+2s+\frac{1}{2}}(x) \\ &\qquad +\sum_s(-1)^s\,b_{k,2s}\sum_{n=0}^{\infty}(-1)^n\,(2n+2s\tfrac{1}{2})\,(2n+1)_{4s}\,J_{2n+2s+\frac{1}{2}}(x) \\ &=\sum_s b_{k,2s-1}\sum_{n=s}^{\infty}(-1)^n\,(2n+\tfrac{1}{2})\,(2n-2s+2)_{4s-2}\,J_{2n+\frac{1}{2}}(x) \\ &\qquad +\sum_s b_{k,2s}\sum_{n=s}^{\infty}(-1)^n\,(2n+\tfrac{1}{2})\,(2n-2s+1)_{4s}\,J_{2n+\frac{1}{2}}(x) \\ &=\sum_{n=0}^{\infty}(-1)^n\,J_{2n+\frac{1}{2}}(x)\sum_s b_{k,\,s}\,(2n+\tfrac{1}{2})\,(2n-s+1)_{2s}. \end{aligned}$$

Now choose $b_{k,s}$ so that

$$(4.3)\qquad x^{2k+1}=\sum_{s=0}^{k}b_{k,s}x(x-s+\tfrac{1}{2})_{2s}.$$

By the Bessel interpolation formula [2, p. 203] the $b_{k,s}$ are given by

$$(4.4)\quad \begin{cases} b_{k,s}=\frac{1}{(2s+1)!}[\Delta^{2s+1}x^{2k+1}]_{x=-s-\frac{1}{2}} \\ \qquad =\frac{1}{(2s+1)!}\sum_{j=0}^{2s+1}(-1)^{j+1}(2s+1_j)(j-s-\frac{1}{2})^{2k+1}. \end{cases}$$

We have therefore

$$(4.5)\quad \begin{cases} \pi^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \, (2n+\tfrac{1}{2})^{2k+1} J_{2n+\frac{1}{2}}(x) \\ \qquad = \sum_{s=1}^{[(k+1)/2]} (-1)^s \, 2^{4s-1}(2s-1)! \, (\tfrac{1}{2}x)^{2s-\frac{1}{2}} \sin x \\ \qquad\qquad + \sum_{s=0}^{[k/2]} (-1)^s \, 2^{4s}(2s)! \, b_{k,2s} (\tfrac{1}{2}x)^{2s+\frac{1}{2}} \cos x. \end{cases}$$

This result may be compared with Va and Vb of [4]. In a similar way we can sum the series

$$\sum_{n=0}^{\infty} (-1)^n \, (2n+(3/2))^{2k+1} J_{2n+3/2)}(x).$$

5. In the next place we take

$$\begin{aligned} (\tfrac{1}{2}x)^k J_\nu(x) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(\nu+r+1)} (\tfrac{1}{2}x)^{\nu+k+2r} \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(\nu+r+1)} \sum_{s=0}^{\infty} (\nu+k+2r+2s) \frac{\Gamma(\nu+k+2r+s)}{s!} \cdot \\ &\qquad\qquad \cdot J_{\nu+k+2r+2s}(x) \\ &= \sum_{n=0}^{\infty} (\nu+k+2n) \frac{(\nu+1)_{k+n-1}}{n!} J_{\nu+k+2n}(x) \cdot \sum_{r=0}^{n} \frac{(-n)_r(\nu+k+n)_r}{r!(\nu+1)_r}. \end{aligned}$$

Since

$$\sum_{r=0}^{n} \frac{(-n)_r(\nu+k+n)_r}{r!(\nu+1)_r} = \frac{(1-k-n)_n}{(\nu+1)_n} = (-1)^n \frac{(k)_n}{(\nu+1)_n},$$

we get

$$(5.1)\quad \begin{cases} (\tfrac{1}{2}x)^k J_\nu(x) = \dfrac{1}{(k-1)!} \sum_{n=0}^{\infty} (-1)^n(\nu+k+2n)(n+1)_{k-1}(\nu+n+1)_{k-1} \cdot \\ \qquad\qquad \cdot J_{\nu+k+2n}(x). \end{cases}$$

We now consider

$$\begin{aligned} &\sum_s (-1)^s(2s-1)! \, c_{k,2s} (\tfrac{1}{2}x)^{2s} J_\nu(x) + \sum_s (-1)^s(2s)! \, c_{k,2s+1} (\tfrac{1}{2}x)^{2s+1} J_{\nu-1}(x) \\ &\quad = \sum_s (-1)^s c_{k,2s} \sum_{n=0}^{\infty} (-1)^n \, (\nu+2s+2n) \, (n+1)_{2s-1} \, (\nu+m+1)_{2s-1} J_{\nu+2s+2n}(x) \\ &\qquad + \sum_s (-1)^s c_{k,2s+1} \sum_{n=0}^{\infty} (-1)^n \, (\nu+2s+2n) \, (n+1)_{2s} \, (\nu+n)_{2s} J_{\nu+2s+2n}(x) \\ &\quad = \sum_{n=0}^{\infty} (-1)^n \, (\nu+2n) J_{\nu+2n}(x) \cdot \{ \sum_s c_{k,2s} (n-s+1)_{2s-1}(\nu+n-s+1)_{2s-1} \\ &\qquad + \sum_s c_{k,2s+1} (n-s+1)_{2s}(\nu+m-s)_{2s} \}. \end{aligned}$$

We shall try to choose the coefficients $c_{k,s}$ so that

$$(5.2)\quad \begin{cases} x^{2k} = \sum_s c_{k,2s-1} \, (x-\alpha-s+1)_{2s-1}(x+\alpha-s+1)_{2s-1} \\ \qquad\qquad + \sum c_{k,2s}(x-\alpha-s+1)_{2s}(x+\alpha-s)_{2s}, \end{cases}$$

where $\nu = 2\alpha$.

If we assume that (5.2) is satisfied it is clear that

$$\left(\frac{\nu}{2}+n\right)^{2k} = \sum_s c_{k,\,2s-1}(n-s+1)_{2s-1}(\nu+n-s+1)_{2s-1} + \sum_s c_{k,\,2s}(n-s+1)_{2s}(\nu+n-s)_{2s},$$

so that

$$(5.3)\quad \begin{cases} \sum\limits_{n=0}^{\infty}(-1)^n\,(\nu+2n)^{2k+1}\,J_{\nu+2n}(x) \\ \qquad = 2^{2k}\sum\limits_{s=1}^{[k+1/2]}(-1)^s(2s-1)!\;c_{k,2s}\,(\tfrac{1}{2}x)^{2s}\,J_\nu(x) \\ \qquad\quad + 2^{2k}\sum\limits_{s=0}^{[k/2]}(-1)^s(2s)!\;c_{k,2s+1}\,(\tfrac{1}{2}x)^{2s+1}\,J_{\nu-1}(x). \end{cases}$$

In order to get an explicit formula for $c_{k,\,s}$ we notice first that since

$$(x-\alpha-s+1)_{2s}\,(x+\alpha-s)_{2s} = (x^2-(\alpha-s)^2)\,(x-\alpha-s+1)_{2s+1}\,(x+\alpha-s+1)_{2s-1},$$

$$(x-\alpha-s)_{2s+1}\,(x+\alpha-s)_{2s+1} = (x^2-(\alpha+s)^2)\,(x-\alpha-s+1)_{2s}\,(x+\alpha-s)_{2s},$$

(5.2) yields

$$\begin{aligned} x^{2k+2} = \sum_s c_{k,2s-1}\,\{&(x-\alpha-s+1)_{2s}(x+\alpha-s)_{2s} \\ &+(\alpha-s)^2(x-\alpha-s+1)_{2s-1}(x+\alpha-s+1)_{2s-1}\} \\ +\sum_s c_{k,2s}\,\{&(x-\alpha-s)_{2s+1}(x+\alpha-s)_{2s+1} \\ &+(\alpha+s)^2(x-\alpha-s+1)_{2s}(x+\alpha-s)_{2s}\}. \end{aligned}$$

It follows that

$$(5.4)\quad \begin{cases} c_{k+1,\,2s-1} = (\alpha-s)^2]c_{k,\,2s-1}+c_{k,\,2s-2}, \\ c_{k+1,\,2s} = c_{k,\,2s-1}+(\alpha+s)^2 c_{k,\,2s}. \end{cases}$$

Now put

$$f_s(x) = \sum_{k=s}^{\infty} x^k c_{k,\,s}.$$

Then it follows from (5.4) that

$$(5.5)\quad \begin{cases} f_{2s-1}(x) = \dfrac{x}{1-(\alpha-s)^2x}\,f_{2s-2}(x), \\ f_{2s}(x) = \dfrac{x}{1-(\alpha+s)^2x}\,f_{2s-1}(x). \end{cases}$$

Since

$$c_{k,\,0} = \alpha^{2k},\quad f_0(x) = \frac{1}{1-\alpha^2x},$$

it is clear that (5.5) implies

$$(5.6)\quad \begin{cases} f_{2s}(x) = \dfrac{x^{2s}}{\prod\limits_{j=-s}^{s}(1-(\alpha+j)^2x)}, \\ f_{2s-1}(x) = \dfrac{x^{2s-1}}{\prod\limits_{j=-s}^{s-1}(1-(\alpha+j)^2x)}. \end{cases}$$

We shall assume that α is not equal to an integer or half of an odd integer. If we put

$$\frac{x^{2s}}{\prod_{j=-s}^{s}(1-(\alpha+j)^2x)}=\sum_{j=-s}^{s}\frac{A_j}{1-(\alpha+j)^2x},$$

then

$$\begin{aligned}A_j &= \prod_{\substack{i=-s\\ i\neq j}}^{s}((\alpha+j)^2-(\alpha+i)^2)^{-1}\\ &= (-1)^{s-j}\frac{2(\alpha+j)}{(s-j)!(s+j)!}\prod^{s}(2\alpha+i+j)^{-1},\end{aligned}$$

so that

$$c_{k,2s}=\frac{2}{(2s)!}\sum_{j=-s}^{s}(-1)^{s-j}\binom{2s}{s+j}(\alpha+j)^{2k+1}\cdot\prod_{i=-s}^{s}(2\alpha+i+j)^{-1}. \tag{5.7}$$

Similarly if we put

$$\frac{x^{2s-1}}{\prod_{j=-s}^{s-j}(1-(\alpha+j)^2x)}=\sum_{j=-s}^{s-1}\frac{A_j'}{1-(\alpha+j)^2x},$$

then

$$\begin{aligned}A_j' &= \prod_{\substack{i=-s\\ i\neq j}}^{s-1}((\alpha+j)^2-(\alpha+i)^2)^{-1}\\ &=(-1)^{s-j-1}\frac{2(\alpha+j)}{(s-j-1)!\,(s+j)!}\prod_{i=-s}^{s-1}(2\alpha+i+j)^{-1},\end{aligned}$$

so that

$$\left\{\begin{aligned}c_{k,2s-1} &= \frac{2}{(2-1)!}\sum_{j=-s}^{s-1}(-1)^{s-j-1}\binom{2s-1}{s+1}(\alpha+j)^{2k+1}\cdot\\ &\qquad\cdot\prod_{i=-s}^{s-1}(2\alpha+i+j)^{-1}.\end{aligned}\right. \tag{5.8}$$

We see therefore that the sum

$$\sum_{n=0}^{\infty}(-1)^n(\nu+2n)^{2k+1}J_{\nu+2n}(x)$$

is evaluated by means of (5.3), (5.7) and (5.8) with $\nu=2\alpha$. This result may be compared with IIIa and IIIb of [3]. As noted above, it is assumed that ν is not equal to an integer or half of an odd integer.

6. Rutgers has also proved the formula

$$\sum_{n=1}^{\infty}(2n+1)^{2k+1}J_{2n+1}(x)=\sum_{r=0}^{k}(\tfrac{1}{2}x)^{2r+1}\sum_{s=0}^{r}(-1)^s\frac{(2r-2s+1)^{2k+1}}{s!(2r-s+1)!}, \tag{6.1}$$

$$\sum_{n=1}^{\infty}(2n)^{2k}J_{2n}(x)=\sum_{r=1}^{k}(\tfrac{1}{2}x)^{2r}\sum_{s=0}^{r-1}(-1)^s\frac{(2r-2s)^{2k}}{s!(2r-s)}\ (k\geqslant 1). \tag{6.2}$$

See also [1, p. 273].

Since

$$\sum_{s=0}^{2r+1}(-1)^s\binom{2r+1}{s}(2r-2s+1)^{2k+1}=2\sum_{s=0}^{r}(-1)^s\binom{2r+1}{s}(2r-2s+1)^{2k+1},$$

$$\sum_{s=0}^{2r}(-1)^s\binom{2r}{s}(2r-2s)^{2k}=2\sum_{s=0}^{r-1}(-1)^s\binom{2r}{s}(2r-2s)^{2k},$$

it is clear that (6.1) and (6.2) become

$$\sum_{n=0}^{\infty}(2n+1)^{2k+1}J_{2n+1}(x)=\tfrac{1}{2}\sum_{r=0}^{k}2^{2k-2r}b_{k,r}\,x^{2r+1}, \tag{6.3}$$

$$\sum_{n=1}^{\infty}(2n)^{2k}J_{2n}(x)=\tfrac{1}{2}\sum_{r=1}^{k}2^{2k-2r}\,a_{k,r}\,x^{2r}\ (k\geqslant 1) \tag{6.4}$$

where $a_{k,r}$ and $b_{k,r}$ are defined by (2.6) and (4.4), respectively.

For $k=0$ we of course have the familiar expansion

$$J_0(x)+2\sum_{1}^{\infty}J_{2n}(x)=1.$$

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REFERENCES

1. Nielsen, N., Handbuch der Theorie der Cylinderfunktionen, Leipzig, 1904.
2. Nörlund, N. E., Vorlesungen über Differenzenrechnung, Berlin, 1924.
3. Rutgers, J. G., Extension d'une serie des fonctions de Bessel, due a Lommel et de quelques series des fonctions de Bessel analogues, I, Nederl. Akademie van Wetenschappen, Proceedings of the section of sciences, **45**, 929–936 (1942).
4. ———, Extension d'une serie des fonctions de Bessel, due a Lommel, et de quelques series des fonctions de Bessel analogues, II, Nederl. Akademie van Wetenschappen, Proceedings of the section of sciences, **45**, 987–993 (1942).